Kuwait University	Math. 101	Date	: October 19th, 2000
Dept.of Maths.&Comp. Sci.	First Exam.	Duration	: 75 minutes.

Calculators, Mobile Telephones and Pagers are not allowed.
Answer all the following questions. Show your work.

1. Find the following limits, if they exist:

   a) $\lim_{x\to -2} \frac{x^3+8}{x+2}$ (3 pts.)

   b) $\lim_{x\to -\infty} \frac{\sqrt{x^2+x}}{2x+1}$ (3 pts.)

   c) $\lim_{x\to 0} \frac{\sqrt{1+\sin x}-1}{\tan x}$ (3 pts.)

   d) $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x-1}\right)$ (3 pts.)

2. Find the vertical and horizontal asymptotes, if any, of the graph of the function

$$f(x) = \frac{|x|}{2x-3}.$$ (3 pts.)

3. Let k be a real number and

$$f(x) = \begin{cases} \frac{\sqrt{x-k+1}-1}{x-k} & , \text{ if } \quad x > k \\ 2x^2 & , \text{ if } \quad x \le k . \end{cases}$$

Find all values of k so that f is continuous on $(-\infty, \infty)$. (3 pts.)

4. Classify the discontinuities of f as removable, jump, or infinite, where

$$f(x) = \frac{|x-1|\,(x-2)}{(x-1)\,(x^2-5x+6)}$$ (3 pts.)

5. a) State the Intermediate Value Theorem. (1 pt.)

   b) Use the intermediate value theorem to show that the graphs of the equations $y = x^3$ and $y = x+1$ intersect. (3 pts.)

Good Luck

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1. (a) $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2} = \lim\limits_{x\to -2} \dfrac{(x+2)(x^2-2x+4)}{x+2} = \boxed{12}$

(b) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2+x}}{2x+1} = \lim\limits_{x\to -\infty} \dfrac{|x|\sqrt{1+\left(\frac{1}{x}\right)}}{x\left(2+\frac{1}{x}\right)} = \lim\limits_{x\to -\infty} \dfrac{-x\sqrt{1+\left(\frac{1}{x}\right)}}{x\left(2+\frac{1}{x}\right)} = \boxed{-\frac{1}{2}}$

(c) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x}-1}{\tan x} \times \dfrac{\sqrt{1+\sin x}+1}{\sqrt{1+\sin x}+1} = \lim\limits_{x\to 0} \dfrac{\sin x}{\left(\frac{\sin x}{\cos x}\right)\left(\sqrt{1+\sin x}+1\right)} = \boxed{\frac{1}{2}}$

(d) $-1 \le \sin\left(\dfrac{1}{x-1}\right) \le 1 \ \forall x \in \mathbb{R}-\{1\} \Rightarrow$

$-(x-1)^2 \le (x-1)^2 \sin\left(\dfrac{1}{x-1}\right) \le (x-1)^2 \ \forall x \in \mathbb{R}-\{1\}.$

$\lim\limits_{x\to 1} (x-1)^2 = 0 = \lim\limits_{x\to 1} -(x-1)^2$. From The Sandwich theorem:

$\lim\limits_{x\to 1} (x-1)^2 \sin\left(\dfrac{1}{x-1}\right) = 0.$

2. $2x-3=0 \Rightarrow x=\frac{3}{2}$.

$\lim\limits_{x\to \frac{3}{2}^+} f(x) = \infty, \ \lim\limits_{x\to \frac{3}{2}^-} f(x) = -\infty \Rightarrow \boxed{x=\frac{3}{2}}$ is V.A. for the graph of f.

$\lim\limits_{x\to \infty} f(x) = \frac{1}{2} \Rightarrow \boxed{y=\frac{1}{2}}$ is H.A. for the graph of f.

$\lim\limits_{x\to -\infty} f(x) = -\frac{1}{2} \Rightarrow \boxed{y=-\frac{1}{2}}$ is H.A. for the graph of f.

3. $f(k) = 2k^2 = \lim\limits_{x\to k^-} f(x)$

$\lim\limits_{x\to k^+} f(x) = \lim\limits_{x\to k^+} \dfrac{\sqrt{x-k+1}-1}{x-k} \times \dfrac{\sqrt{x-k+1}+1}{\sqrt{x-k+1}+1} = \boxed{\frac{1}{2}}$

f is continuous at k when $2k^2 = \frac{1}{2} \Rightarrow \boxed{k=\pm\frac{1}{2}}$

4. $f(x) = \dfrac{|x-1|(x-2)}{(x-1)(x-2)(x-3)}$

$\lim\limits_{x\to 1^+} f(x) = -\frac{1}{2}, \ \lim\limits_{x\to 1^-} f(x) = \frac{1}{2} \Rightarrow$ The graph of f has **jump** discontinuity at $\boxed{x=1}$.

$f(2)$ is undefined and $\lim\limits_{x\to 2} f(x) = -1 \Rightarrow$The graph of f has **removable** discontinuity at $\boxed{x=2}$.

$\lim\limits_{x\to 3^+} f(x) = \infty, \ \lim\limits_{x\to 3^-} f(x) = -\infty \Rightarrow$The graph of f has **infinite** discontinuity at $\boxed{x=3}$.

5. b) $x^3 = x+1$. If the equation $x^3-x-1=0$ has a real solution then the two graphs intersect. Let $f(x) = x^3-x-1$.

$f(0) = -1 < 0$, $f(2) = 5 > 0$. Since f is continuous on $[0,2]$ (polynomial function) then from the I.V.T. there exist at least one $c \in (0,2)$ such that $f(c) = 0$.